
THE BINOMIAL THEOREM – PASCAL'S TRIANGLE

Imagine expanding $(a + b)^{100}$ using regular algebra. This chapter will show us a neat shortcut for solving this type of problem. The concepts involved in raising a binomial to a high power have applications in probability, from the flipping of coins to quality control in manufacturing light bulbs.



❑ **EXPANDING AND SEEING PATTERNS**

The goal of the Binomial Theorem is to expand powers of binomials, things like $(a + b)^{100}$, without actually multiplying anything out. Let's start with some expansions we already know how to do using brute-force algebra.

$$(a + b)^0 = 1 \quad \text{(anything [but 0] to the zero power is 1)}$$

$$(a + b)^1 = a + b \quad \text{(anything to the 1st power is itself)}$$

$$(a + b)^2 = a^2 + 2ab + b^2 \quad \text{(just multiply } a + b \text{ by } a + b)$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \quad \text{(multiply } a^2 + 2ab + b^2 \text{ by } a + b)$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \quad \text{(etc.)}$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \quad \text{(etc.)}$$

Homework

1. Verify the calculations above for each expansion:

$$(a + b)^2 \qquad (a + b)^3 \qquad (a + b)^4 \qquad (a + b)^5$$

2. How many terms are there in each expansion?

$$\begin{array}{lll} \text{a. } (a + b)^0 & \text{b. } (a + b)^1 & \text{c. } (a + b)^2 \\ \text{d. } (a + b)^3 & \text{e. } (a + b)^4 & \text{f. } (a + b)^5 \end{array}$$

3. a. How many terms are there in the expansion of $(a + b)^{100}$?
 b. How many terms are there in the expansion of $(a + b)^n$?
4. a. Notice the term $10a^3b^2$ in the expansion of $(a + b)^5$. What is the *sum* of the exponents on the a and the b ?
 b. In the same expansion, the 5th term is $5ab^4$. What is the *sum* of the exponents on the a and the b ?
5. a. See the first term in the expansion of $(a + b)^4$? It's a^4 . What is the *sum* of the exponents on the a and the b ? [Hint: Maybe there's no explicit b in the term a^4 , but what power of b could be placed next to the a^4 so that it remains a^4 ? That is, $a^4b^? = a^4$.]
 b. What is the *sum* of the exponents for any term in the expansion of $(a + b)^4$?
6. What is the *sum* of the exponents for any term in the expansion of $(a + b)^n$?
7. If the expansion of $(a + b)^k$ has a term of the form $ca^{44}b^{33}$, where c is a constant, what is the value of exponent k ?

□ OBSERVATIONS ON THE EXPANSIONS

1. The **number of terms** in each expansion is one more than the exponent on the binomial. For example, the expansion of $(a + b)^5$ has **6** terms, and the expansion of $(a + b)^n$ has **$n + 1$** terms.
2. Within any given term, the **sum of the exponents** always equals the exponent on the binomial. For instance, looking at the expansion of $(a + b)^5$ from page 1:

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

If we “patch up” the expansion of $(a + b)^5$ to include both the a and the b in every term, we can write the previous expansion as

$$(a + b)^5 = a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + a^0b^5$$

We notice that in any of the 6 terms, the sum of the exponents is always **5**. This always works: In every term in the expansion of $(a + b)^n$, the sum of the exponents is always **n** .

3. The exponents on the a go down while the exponents on the b go up. Again, look at the form in the previous observation:

$$(a + b)^5 = a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + a^0b^5$$

Notice that the exponents on the a go down: 5, 4, 3, 2, 1, 0, while the exponents on the b go up: 0, 1, 2, 3, 4, 5.

EXAMPLE 1: Apply the three observations above to the expansion of $(a + b)^{100}$.

Solution: First, there will be **101** terms in the expansion. Second, in any term the sum of the exponents will be **100**. For instance, the term with a^{40} in it will also have b^{60} in it. Third, the exponents on the a will go 100, 99, 98, . . . , 2, 1, 0 while the exponents on the b will go 0, 1, 2, . . . , 98, 99, 100. In short, using boxes for the unknown coefficients (the front numbers), the expansion of $(a + b)^{100}$ looks like this:

$$\square a^{100} + \square a^{99}b + \square a^{98}b^2 + \cdots + \square a^2b^{98} + \square ab^{99} + \square b^{100}$$

Homework

8. Consider the expansion of $(a + b)^{23}$. How many terms are there in the expansion? In the term containing a^{15} , what is the exponent on the b ?
9. There are 102 terms in the expansion of $(x + y)^n$. What is n ?
10. Find “Row 10” of Pascal’s Triangle.

□ ***AN EXAMPLE OF THE BINOMIAL THEOREM***

Let’s use the three observations mentioned earlier, along with Pascal’s Triangle, to deduce the expansion, without using algebra, of

$$(a + b)^7$$

1. There will be 8 terms in the expansion.
2. The sum of the exponents in each term will be 7.
3. The exponents on the a will go from 7 down to 0, while those on the b will go from 0 up to 7.

Also, the relevant row of Pascal’s Triangle is

$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$

There are two ways we can know this. First, it’s the only row in Pascal’s Triangle with 8 entries in it, and we need 8 coefficients for the 8 terms of the expansion. Moreover, the second number in the row, the 7, matches the exponent in the expression $(a + b)^7$.

Let's try it: $(a + b)^7$

$$= \underline{1}a^7b^0 + \underline{7}a^6b^1 + \underline{21}a^5b^2 + \underline{35}a^4b^3 + \underline{35}a^3b^4 \\ + \underline{21}a^2b^5 + \underline{7}a^1b^6 + \underline{1}a^0b^7$$

The underlined coefficients come from Row 7 of Pascal's Triangle.

$$= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

Is the Binomial Theorem worth knowing? Try expanding $(a + b)^7$ without it.

Homework

11. Find the first two terms and the last two terms of $(a + b)^{100}$.
12. Even though we know how to expand $(a + b)^2$ from basic algebra, show that the Binomial Theorem produces the same result.
13. Expand $(m + z)^8$.
14. Expand $(x + y)^{10}$.

Review Problems

15. How many terms are there in the expansion of $(a + b)^{200}$?
16. What is the sum of the exponents for any term in the expansion of $(a + b)^{123}$?

17. Consider the expansion of $(a + b)^{250}$.
- How many terms are there in the expansion?
 - In the term containing a^{100} , what is the exponent on the b ?
 - What is the first term of the expansion?
 - What is the second term of the expansion?
 - What is the second-to-last term of the expansion?
 - What is the last term of the expansion?
18. Expand $(u + w)^9$.

Solutions

1. They're boring, but do them!
2. a. 1 b. 2 c. 3 d. 4 e. 5 f. 6
3. a. 101 b. $n + 1$ 4. a. $3 + 2 = 5$ b. 5
5. a. 4 b. 4 6. n
7. $k = 77$ 8. 24; 8 9. $n = 101$
10. 1 10 45 120 210 252 210 120 45 10 1
11. $a^{100} + 100a^{99}b + \cdots + 100ab^{99} + b^{100}$

- 12.** To expand $(a + b)^2$ by the Binomial Theorem, we know that the relevant row of Pascal's Triangle is "1 2 1". We also know that the exponents on the a go from 2 down to 0, while the exponents on the b go from 0 up to 2. Putting it all together:

$$(a + b)^2 = 1a^2b^0 + 2a^1b^1 + 1a^0b^2 = a^2 + 2ab + b^2, \text{ as it should be.}$$

13. $m^8 + 8m^7z + 28m^6z^2 + 56m^5z^3 + 70m^4z^4 + 56m^3z^5 + 28m^2z^6 + 8mz^7 + z^8$

14. $x^{10} + 10x^9y + 45x^8y^2 + 120x^7y^3 + 210x^6y^4 + 252x^5y^5 + 210x^4y^6 + 120x^3y^7 + 45x^2y^8 + 10xy^9 + y^{10}$

15. 201 **16.** 123

17. a. 251 b. 150 c. a^{250} d. $250a^{249}b$
e. $250ab^{249}$ f. b^{250}

18. $u^9 + 9u^8w + 36u^7w^2 + 84u^6w^3 + 126u^5w^4 + 126u^4w^5 + 84u^3w^6 + 36u^2w^7 + 9uw^8 + w^9$

"Cherish your visions and your dreams, as they are the children of your soul, the blueprints of your ultimate achievements."

– Napoleon Hill

